False Narratives and Political Mobilization: A Dynamic Convergence Result*

Kfir Eliaz, Simone Galperti, and Ran Spiegler†

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In this supplementary document, we consider a simple and natural dynamic process that determines which platforms garner maximal support over time. We show that the process converges to the unique equilibrium distribution over policies and coalitions in our main result. This global convergence result provides a dynamic foundation for our equilibrium concept.

Time is discrete and denoted by $t = 1, 2, \ldots$. In each period t, there is a distribution σ_t over platforms (a, C, S), where $a \in \{\ell, h\}$, $C \subseteq N$, and $S \in \mathcal{S}$. Let the initial σ_1 be any distribution with full support over the set of platforms using admissible coalitions. Since the set of platforms is finite, this distribution is well-defined. The distribution σ_t evolves according to the following adjustment. For every $t \geq 2$, let

$$\overline{(a, C, S)}_t \in \arg\max_{(a', C', S')} U_{\sigma_t}(a', C', S'),$$

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†Eliaz: Tel Aviv University and the University of Utah. Galperti: UC San Diego. Spiegler: Tel Aviv University and University College London.

where ties can be broken arbitrarily. Then, let

$$\sigma_{t+1}(a,C,S) = \begin{cases} \frac{1}{t+1} + \frac{t}{t+1}\sigma_t(a,C,S) & \text{if } (a,C,S) = \overline{(a,C,S)}_t \\ \\ \frac{t}{t+1}\sigma_t(a,C,S) & \text{otherwise.} \end{cases}$$

Thus, for t large enough, we can essentially view $\sigma_t(a,S,C)$ as the empirical frequency with which platform (a,C,S) has been dominant in the available history of data.

Proposition 1. *Every limit point σ of the process σ_t induces the same distribution over policy-coalition pairs (a,C) as that induced by the unique essential equilibrium σ^*.*

This result formalizes and generalizes the dynamic convergence process we discussed in the context of the two-group specification in Section 3.

Proof of Proposition 1

In this proof, we denote platforms by z whenever convenient to simplify notation. For every t, let $\bar{z}_t = (\bar{a}_t, \bar{C}_t, \bar{S}_t) \in \arg\max_z U_{\sigma_t}(z)$ be the dominant platform at period t and let $\overline{U}_{\sigma_t} = U_{\sigma_t}(\bar{z}_t)$ be the payoff it generates. Note that if there exists T such that $\bar{z}_t \neq (a,C,S)$ for all $t \geq T$, then $\sigma_t(a,C,S) \to 0$ as $t \to \infty$. Recall that $U^* = q \cdot F(N^g, g) > 0$. The proof proceeds stepwise.

Step 1. *$\overline{U}_{\sigma_t} \geq U^*$ for every t.*

Proof. Since σ_1 has full support, $\sigma_t(N^g, g, \{0\}) > 0$ for every finite t; therefore, $\overline{U}_{\sigma_t} \geq U_{\sigma_t}(N^g, g, \{0\}) = U^*$ for every t. □

Step 2. *If $\bar{z}_t = (h,C,S)$, then $C = N^h$ and $U_{\sigma_t}(h,C,S) = U^*$.*

Proof. For every platform (h,C,S) such that $C \subset N^h$, $U_{\sigma_t}(h,C,S) < U_{\sigma_t}(N^g, g, \{0\})$ because $Pr_{\sigma_t}(y = 1 \mid x_S(h,C)) \leq q$ and $F(h,C) < F(N^g, g)$. This also implies that $U_{\sigma_t}(N^g, g, S) \leq U^*$ for all S and hence the last equality. □

Step 3. *For all t, there exists $t' > t$ such that $\bar{z}_{t'} = (N^g, g, S)$ for some S.*

Proof. Step 1 implies that

$$\liminf_{t\to\infty} \overline{U}_{\sigma_t} \geq U^*.$$

Suppose there exists t such that $\bar{z}_{t'} = (\ell, \bar{C}_{t'}, \bar{S}_{t'})$ for all $t' \geq t$. This implies that $Pr_{\sigma_t}(y = 1 \mid x_{\bar{S}_t}(\bar{a}_t, \bar{C}_t)) \to 0$, which is inconsistent with $\liminf_{t\to\infty} \overline{U}_{\sigma_t} > 0$. □

Step 4. $\liminf \overline{U}_{\sigma_t} = U^*$.

Proof. We have already established that $\liminf_{t\to\infty} \overline{U}_{\sigma_t} \geq U^*$. Note that, if $\overline{U}_{\sigma_t} > U^*$, then $\bar{z}_t = (\ell, C, S)$ for some C and S, because $U_{\sigma_t}(h, C', S') \leq U^*$ for all C' and S'. Now suppose $\liminf_{t\to\infty} \overline{U}_{\sigma_t} > U^*$. Then, there exists T such that for all $t \geq T$, $\bar{z}_t$ involves policy $a = \ell$. This contradicts Step 3. □

Recall that

$$Pr_{\sigma_t}(y = 1 \mid x_S(a, C)) = q \cdot \frac{\sum_{C', S' \mid x_S(h, C') = x_S(a, C)} \sigma_t(h, C', S')}{\sum_{a', C', S' \mid x_S(a', C') = x_S(a, C)} \sigma_t(a', C', S')}$$

Step 5. *If $\bar{z}_t = (N^g, g, \hat{S})$ and $x_S(N^g, g) = x_S(\ell, C)$, then*

$$Pr_{\sigma_{t+1}}(y = 1 \mid x_S(\ell, C)) > Pr_{\sigma_t}(y = 1 \mid x_S(\ell, C))$$

Proof. Given $\bar{z}_t = (N^g, g, \hat{S})$, for every (ℓ, C, S) such that $x_S(N^g, g) = x_S(\ell, C)$,

$$\begin{aligned}
Pr_{\sigma_{t+1}}(y = 1 \mid x_S(\ell, C)) &= q\frac{\frac{1}{t+1} + \frac{t}{t+1}\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h, C', S')}{\frac{1}{t+1} + \frac{t}{t+1}\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a', C', S')} \\
&= q\frac{\frac{1}{t} + \sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h, C', S')}{\frac{1}{t} + \sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a', C', S')} \\
&> q\frac{\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h, C', S')}{\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a', C', S')} \\
&= Pr_{\sigma_t}(y = 1 \mid x_S(\ell, C))
\end{aligned}$$

□

Step 6. *If $\bar{z}_t = (\ell, \hat{C}, \hat{S})$, then for every (ℓ, C, S),*

$$Pr_{\sigma_{t+1}}(y = 1 \mid x_S(\ell, C)) \leq Pr_{\sigma_t}(y = 1 \mid x_S(\ell, C))$$

with strict inequality if and only if $x_S(\ell, \hat{C}) = x_S(\ell, C)$.

Proof. If $\bar{z}_t = (\ell, \hat{C}, \hat{S})$ and $x_S(\ell, \hat{C}) \neq x_S(\ell, C)$, then by definition, $Pr_{\sigma_{t+1}}(y = 1 \mid x_S(\ell, C)) = Pr_{\sigma_t}(y = 1 \mid x_S(\ell, C))$. Now suppose that $\bar{z}_t = (\ell, \hat{C}, \hat{S})$ and $x_S(\ell, \hat{C}) = x_S(\ell, C)$. Then,

$$\begin{aligned}
Pr_{\sigma_{t+1}}(y = 1 \mid x_S(\ell, C)) &= q\frac{\frac{t}{t+1}\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h, C', S')}{\frac{1}{t+1} + \frac{t}{t+1}\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a', C', S')} \\
&= q\frac{\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h, C', S')}{\frac{1}{t} + \sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a', C', S')} \\
&< q\frac{\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h, C', S')}{\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a', C', S')} \\
&= Pr_{\sigma_t}(y = 1 \mid x_S(\ell, C))
\end{aligned}$$

□

Step 7. *If (ℓ, C, S) is such that $x_S(\ell, C) \neq x_S(N^g, g)$, then $\sigma_t(\ell, C, S) \to 0$*

as $t \to \infty$.

Proof. Suppose $\sigma_t(\ell, C, S) \not\to 0$. Then, there exists a subsequence such that $\sigma_t(\ell, C, S) \to \hat{\sigma} > 0$, which implies that the denominator of $Pr_{\sigma_t}(y = 1|x_S(\ell, C))$ converges to a strictly positive number along the subsequence. However, the numerator of $Pr_{\sigma_t}(y = 1|x_S(\ell, C))$ converges to zero by Step 2, because $\sigma_t(h, C', S') \to 0$ if $x_S(h, C') = x_S(\ell, C)$ and hence C'^h. Therefore, $U_{\sigma_t}(\ell, C, S) \to 0$ along the subsequence, which contradicts $\sigma_t(\ell, C, S) \to \hat{\sigma} > 0$. $\square$

Step 8. *If (ℓ, C, S) is such that $x_S(\ell, C) = x_S(N^g, g)$, then*

$$\liminf_{t\to\infty} \sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h, C', S') = \liminf_{t\to\infty} \sum_{S'} \sigma_t(N^g, g, S') \equiv \underline{\sigma} > 0$$

Proof. The first equality follows because $\sigma_t(h, C', S') \to 0$ if C'^h by Step 2 and because $x_S(\ell, C) = x_S(N^g, g)$. The last inequality is strict because, if $\underline{\sigma} = 0$, there exists a subsequence such that $\sum_{C',S'} \sigma_t(h, C', S') \to 0$ and hence $\sigma_t(\ell, C, S) \to \hat{\sigma} > 0$ for some (ℓ, C, S) such that $x_S(\ell, C) = x_S(N^g, g)$. However, in this case there exists T such that for all $t \geq T$ in this subsequence the numerator of $Pr_{\sigma_t}(y = 1 \mid x_S(\ell, C))$ becomes arbitrarily small and hence $U_{\sigma_t}(\ell, C, S) < U^*$, which is inconsistent with $\hat{\sigma} > 0$. $\square$

Step 9. $\limsup_{t\to\infty} \overline{U}_{\sigma_t} \leq U^*$.

Proof. Suppose $\limsup_{t\to\infty} \overline{U}_{\sigma_t} = \bar{U} > U^*$. Let

$$\bar{P} = \left\{ (\ell, C, S) \mid \limsup_{t\to\infty} U_{\sigma_t}(\ell, C, S) = \bar{U} \right\},$$

which must be non-empty because the set of platforms is finite. Note that $(\ell, C, S) \in \bar{P}$ only if $x_S(\ell, C) = x_S(N^g, g)$. By finiteness of $\bar{P}$, there exists a common subsequence, T, and $\varepsilon > 0$ such that for all $t' \geq T$ in this subsequence $U_{\sigma_{t'}}(\ell, C, S) \geq U^* + \varepsilon$ for all $(\ell, C, S) \in \bar{P}$. By Step 3, there must exist a $t > T$ (not necessarily in the subsequence) such that $\bar{z}_t = (N^g, g, S)$ and hence $\overline{U}_{\sigma_t} = U^*$. Therefore, $U_{\sigma_t}(\ell, C, S) \leq U^*$ for all $(\ell, C, S) \in \bar{P}$. By Step 5, for all $(\ell, C, S) \in \bar{P}$,

$$\begin{aligned}
\frac{U_{\sigma_{t+1}}(\ell, C, S)}{U_{\sigma_t}(\ell, C, S)} &= \frac{\left(\frac{\frac{1}{t}+\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h,C',S')}{\frac{1}{t}+\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a',C',S')}\right)}{\left(\frac{\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h,C',S')}{\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a',C',S')}\right)} \\
&< \frac{\left(\frac{\frac{1}{t}+\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h,C',S')}{\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a',C',S')}\right)}{\left(\frac{\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h,C',S')}{\sum_{a',C',S'|x_S(a',C')=x_S(\ell,C)} \sigma_t(a',C',S')}\right)} \\
&= \frac{\frac{1}{t}}{\sum_{C',S'|x_S(h,C')=x_S(\ell,C)} \sigma_t(h,C',S')} + 1
\end{aligned}$$

which converges to 1 as $t \to \infty$ by Step 8. Therefore, for every $\delta > 0$, we can pick T large enough such that, for all $t \geq T$ such that $\bar{z}_t = (h, C, S)$,

$$\frac{U_{\sigma_{t+1}}(\ell, C, S)}{U_{\sigma_t}(\ell, C, S)} \leq 1 + \delta$$

for all $(\ell, C, S) \in \bar{P}$. Finally, this means that we can also pick T and $t \geq T$ so that $\bar{z}_t = (h, C, S)$ and $U_{\sigma_{t+1}}(\ell, C, S) < U^* + \varepsilon$ for all $(\ell, C, S) \in \bar{P}$. Therefore, $U_{\sigma_{t+k}}(\ell, C, S) < U^* + \varepsilon$ for all $(\ell, C, S) \in \bar{P}$ and all $k \geq 1$, because by Step 6 the payoff of (ℓ, C, S) is weakly decreasing when $U_{\sigma_t}(\ell, C, S) > U^*$. We, thus, reach a contradiction. $\square$

Steps 4 and 9 imply that $\lim_{t \to \infty} \overline{U}_{\sigma_t} = U^*$. Now, denote by Σ the set of limit points of σ_t.

Step 10. *All $\sigma \in \Sigma$ must induce the same joint distribution over (a, C), and this distribution must coincide with the unique equilibrium distribution.*

Proof. Note that $U_\sigma(z)$ is continuous in σ for all z. The previous conclusion implies that, for every $\sigma \in \Sigma$ and every z, $U_\sigma(z) \leq U^*$, with equality for

$z \in Supp(\sigma)$. The equilibrium characterization results in Sections 3 and 4 established that every σ that satisfies this property induces the same distribution over (a, C). This completes the proof. $\square$