A Model of Competing Narratives: Correction of the Proof of Claim 2*

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This note corrects and improves the proof of Claim 2 in Section 5 of our paper "A model of competing narratives" (*AER* 2020). The last part of the original proof contained a few errors.

To simplify exposition, we consider the $\varepsilon \to 0$ limit and thus effectively set $\varepsilon = 0$ throughout the proof. (In principle, it would have been more rigorous to carry ε through the steps and take the $\varepsilon \to 0$ limit after the relevant expressions are derived. This would lead to the same result.)

Let σ be an equilibrium, and use the shorthand notation $\alpha_\theta = \alpha_\theta(\sigma)$. Let us calculate $p_G(y = 1 \mid a, \theta)$ for each of the four available narratives:

$$
\begin{aligned}
p_{G^{RE}}(y &= 1 \mid a, \theta) = p(y = 1 \mid a, \theta) = \frac{1}{2}(a + \theta) \\
p_{G^{n}}(y &= 1 \mid a, \theta) = p(y = 1) = \frac{1}{2}[\delta(1 + \alpha_1) + (1 - \delta)\alpha_0] \\
p_{G^{d}}(y &= 1 \mid a, \theta) = p(y = 1 \mid \theta) = \frac{1}{2}(\alpha_\theta + \theta) \\
p_{G^{e}}(y &= 1 \mid a, \theta) = p(y = 1 \mid a) = \frac{1}{2}[a + p(\theta = 1 \mid a)]
\end{aligned}
$$

where

$$
\begin{aligned}
p(\theta &= 1 \mid a = 1) = \frac{\delta\alpha_1}{\delta\alpha_1 + (1 - \delta)\alpha_0} \\
p(\theta &= 1 \mid a = 0) = \frac{\delta(1 - \alpha_1)}{\delta(1 - \alpha_1) + (1 - \delta)(1 - \alpha_0)}
\end{aligned}
$$

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It follows that the net anticipatory utility induced by a policy d coupled with any of the four narratives is:

$$\begin{aligned}
U(G^{RE}, d \quad | \quad \theta) &= \frac{1}{2}\theta + \frac{1}{2}d - C(d) \\
U(G^{n}, d \quad | \quad \theta) &= \frac{1}{2}[\delta(1+\alpha_1) + (1-\delta)\alpha_0] - C(d) \\
U(G^{d}, d \quad | \quad \theta) &= \frac{1}{2}(\alpha_\theta + \theta) - C(d) \\
U(G^{e}, d \quad | \quad \theta) &= \frac{1}{2}d - C(d) + \frac{1}{2}\left[\frac{\delta\alpha_1 d}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{\delta(1-\alpha_1)(1-d)}{\delta(1-\alpha_1) + (1-\delta)(1-\alpha_0)}\right]
\end{aligned}$$

Let us begin with a few preliminary observations regarding the policies that must accompany each of the four possible narratives in any equilibrium. First, the policy that maximizes net anticipatory utility under G^d or G^n is $d^* = 0$. Therefore, if any of these narratives prevails in some state, it must be coupled with $d = 0$. Second, the policy that maximizes net anticipatory utility under G^{RE} is by definition d^{RE}. Therefore, if this narrative prevails in some state, it must be coupled with d^{RE}. Finally, as to the narrative G^e, note that the term

$$\frac{\delta\alpha_1 d}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{\delta(1-\alpha_1)(1-d)}{\delta(1-\alpha_1) + (1-\delta)(1-\alpha_0)} \tag{1}$$

is strictly increasing (decreasing) in d whenever $\alpha_1 > \alpha_0$ ($\alpha_1 < \alpha_0$). It follows that the policy d^e that maximizes net anticipatory utility under G^e satisfies $d^e > d^{RE}$ ($d^e < d^{RE}$) whenever $\alpha_1 > \alpha_0$ ($\alpha_1 < \alpha_0$). Since $C'(1) > 1$, d^{RE} and d^e are both strictly below 1. Therefore, $\alpha_\theta < 1$ for all θ.

We now characterize the equilibrium distribution in each state. First, consider the realization $\theta = 1$. Then,

$$U(G^{RE}, d^{RE} \mid \theta = 1) = \frac{1}{2}(1 + d^{RE}) - C(d^{RE}) = \frac{1}{2} + \max_d[\frac{1}{2}d - C(d)] > \frac{1}{2}$$

For any $\alpha_0, \alpha_1 \in [0,1]$ and $d \in [0,1)$,

$$\frac{\delta\alpha_1 d}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{\delta(1-\alpha_1)(1-d)}{\delta(1-\alpha_1) + (1-\delta)(1-\alpha_0)} < 1 \tag{2}$$

Therefore,

$$U(G^e, d \mid \theta = 1) < U(G^{RE}, d \mid \theta = 1)$$

for any $d \in [0,1)$, and hence, G^e cannot be a prevailing narrative in $\theta = 1$. In addition, a simple calculation establishes that

$$U(G^d, 0 \mid \theta = 1) > U(G^n, 0 \mid \theta = 1)$$

Therefore, G^n cannot be a prevailing narrative in $\theta = 1$. It follows that the only candidates for prevailing narratives in $\theta = 1$ are G^{RE} and G^d.

Suppose $Supp(\sigma_1) = \{(G^d, 0)\}$. Then, $\alpha_1 = 0$, which implies

$$U(G^d, 0 \mid \theta = 1) = \frac{1}{2} < U(G^{RE}, d^{RE} \mid \theta = 1)$$

a contradiction. Now suppose $Supp(\sigma_1) = \{(G^{RE}, d^{RE})\}$. Then, $\alpha_1 = d^{RE}$, in which case

$$U(G^d, 0 \mid \theta = 1) = \frac{1}{2}(d^{RE} + 1) > U(G^{RE}, d^{RE} \mid \theta = 1)$$

a contradiction. The only remaining case is that $Supp(\sigma_1) = \{(G^d, 0), (G^{RE}, d^{RE})\}$. Then,

$$U(G^{RE}, d^{RE} \mid \theta = 1) = U(G^d, 0 \mid \theta = 1)$$

which implies

$$\alpha_1 = d^{RE} - 2C(d^{RE}) \tag{3}$$

This completes the characterization of σ_1. Note that it is independent of σ_0.

Next, consider the realization $\theta = 0$. For any d,

$$U(G^e, d \mid \theta = 0) - U(G^{RE}, d \mid \theta = 0) = \frac{1}{2}\left[\frac{\delta\alpha_1 d}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{\delta(1-\alpha_1)(1-d)}{\delta(1-\alpha_1) + (1-\delta)(1-\alpha_0)}\right]$$

which is strictly positive since $\alpha_1 \in (0,1)$. Therefore, G^{RE} cannot be a prevailing narrative in $\theta = 0$. Likewise,

$$U(G^n, 0 \mid \theta = 0) > U(G^d, 0 \mid \theta = 0)$$

and hence, G^d cannot be a prevailing narrative in $\theta = 0$. It follows that the only candidates for prevailing narratives in $\theta = 1$ are G^e and G^n.

Let us guess an equilibrium in which $\alpha_0 = \alpha_1$. Then,

$$U(G^e, d \mid \theta = 0) = \frac{1}{2}d - C(d) + \frac{1}{2}\delta$$

and the policy that maximizes it is $d^e = d^{RE}$. Thus, plugging (3), we obtain

$$\begin{aligned} U(G^e, d^e \quad | \quad \theta = 0) &= \frac{1}{2}d^{RE} - C(d^{RE}) + \frac{1}{2}\delta = \frac{1}{2}\alpha_1 + \frac{1}{2}\delta \\ U(G^n, 0 \quad | \quad \theta = 0) &= \frac{1}{2}[\delta(1+\alpha_1) + (1-\delta)\alpha_1] = \frac{1}{2}\alpha_1 + \frac{1}{2}\delta \end{aligned}$$

which is consistent with $\alpha_0 \in (0,1)$.

Our final task is to show that there exists no equilibrium with $\alpha_0 \neq \alpha_1$. Suppose first that $\alpha_1 > \alpha_0$. We saw above that in this case, $d^e > d^{RE}$, hence $d^e > \alpha_1$. If $(G^n, 0) \notin Supp(\sigma_0)$, then $\alpha_0 = d^e > \alpha_1$, a contradiction. If $(G^n, 0) \in Supp(\sigma_0)$, then

$$U(G^e, d^e \mid \theta = 0) = U(G^n, 0 \mid \theta = 0) = \frac{1}{2}[\delta(1+\alpha_1) + (1-\delta)\alpha_0] < \frac{1}{2}(\alpha_1 + \delta) \quad (4)$$

Note that

$$\begin{aligned} U(G^e, d^e \quad | \quad \theta = 0) \geq U(G^e, d^{RE} \mid \theta = 0) \\ = \quad \frac{1}{2}d^{RE} - C(d^{RE}) + \frac{1}{2}\delta \left[\frac{\alpha_1 d^{RE}}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{(1-\alpha_1)(1-d^{RE})}{1-\delta\alpha_1 - (1-\delta)\alpha_0} \right] \end{aligned}$$

By (3), this expression is equal to

$$\frac{1}{2}\alpha_1 + \frac{1}{2}\delta \left[\frac{\alpha_1 d^{RE}}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{(1-\alpha_1)(1-d^{RE})}{1-\delta\alpha_1 - (1-\delta)\alpha_0} \right] \quad (5)$$

Recall that by (3), $\alpha_1 < d^{RE}$. Replacing d^{RE} with α_1 in (5) and using the observation that (1) is strictly increasing in d when $\alpha_1 > \alpha_0$, (5) is strictly above

$$\frac{1}{2}\alpha_1 + \frac{1}{2}\delta \left[\frac{\alpha_1^2}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{(1-\alpha_1)^2}{1-\delta\alpha_1 - (1-\delta)\alpha_0} \right]$$

A little algebra establishes that since $\alpha_1 > \alpha_0$,

$$\frac{\alpha_1^2}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{(1-\alpha_1)^2}{1-\delta\alpha_1 - (1-\delta)\alpha_0} > 1$$

we obtain

$$U(G^e, d^e \mid \theta = 0) > \frac{1}{2}(\alpha_1 + \delta)$$

contradicting (4).

The remaining possibility is that $\alpha_0 > \alpha_1$. We saw that in this case,

$d^{RE} > d^e$. Furthermore, since $d^n = 0$, $d^e \geq \alpha_0$. Therefore, $d^{RE} > \alpha_0 > \alpha_1$. If $(G^e, d^e) \notin Supp(\sigma_0)$, then $\alpha_0 = d^n = 0$, a contradiction. It follows that $(G^e, d^e) \in Supp(\sigma_0)$, which means that

$$U(G^e, d^e \mid \theta = 0) \geq U(G^n, 0 \mid \theta = 0) > \frac{1}{2}(\alpha_1 + \delta) \tag{6}$$

where the right-hand inequality follows from $\alpha_0 > \alpha_1$. Now turn to the expression

$$U(G^e, d^e \mid \theta = 0) = \frac{1}{2}d^e - C(d^e) + \frac{1}{2}\delta\left[\frac{\alpha_1 d^e}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{(1-\alpha_1)(1-d^e)}{1-\delta\alpha_1-(1-\delta)\alpha_0}\right]$$

By the definition of d^{RE} and (3),

$$\frac{1}{2}d^e - C(d^e) < \frac{1}{2}d^{RE} - C(d^{RE}) = \frac{1}{2}\alpha_1$$

A little algebra establishes that

$$\frac{\alpha_1 d^e}{\delta\alpha_1 + (1-\delta)\alpha_0} + \frac{(1-\alpha_1)(1-d^e)}{1-\delta\alpha_1-(1-\delta)\alpha_0} \leq 1$$

since

$$d^e \geq \alpha_0 > \delta\alpha_1 + (1-\delta)\alpha_0$$

It follows that

$$U(G^e, d^e \mid \theta = 0) < \frac{1}{2}(\alpha_1 + \delta)$$

contradicting (6).